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SOME FIXED POINT RESULTS FOR $\mathcal{JHR}$ OPERATOR PAIRS IN C^*-ALGEBRA VALUED MODULAR b-METRIC SPACES VIA C_* CLASS FUNCTIONS WITH APPLICATIONS

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Abstract. In this paper, we introduce C^*- algebra valued modular b-metric spaces. Some common fixed point theorems for $\mathcal{JHR}$ operator pairs with new contractive conditions via C_*-class functions in C^*- algebra valued modular b-metric spaces is given here with examples. Some applications on non linear integral equation and on operator equation is also introduced.

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1. Introduction

In ([2],[6]), Bakhtin and Czerwik introduced a generalised metric space which is named as b-metric space with an important property that a metric space can be extended to b-metric space but converse does not hold. In present time, C^*-algebra is an important research area in quantum mechanics, modern mathematics etc. Many researchers generalise ([1]-[26]) all new studies from metric spaces to b-metric spaces and in C^*-algebra also.

In [17], Ma et al. introduced the concept of C^*-algebra valued metric spaces and generalised this concept in [18] as C^*-algebra valued b-metric spaces. In ([19],[20],[21]) Moeini et al. introduced C^*-algebra valued modular metric spaces based on the papers ([4],[5],[25]) of Chistyakov on modular metric spaces and Shateri on C^*-algebra valued modular spaces. In [13], Ege et al. introduced modular b-metric spaces, a generalisation of modular metric spaces. These kind of studies motivated us to generalise C^*-algebra valued modular metric spaces in fixed point theory.

In this paper, we introduce C^*-algebra valued modular b-metric spaces and noncommuting $\mathcal{JHR}$ operator pairs in this space. Some common fixed point theorems with a new type of contractive conditions via C_* class functions is established here with suitable examples. Applications for the existence and uniqueness results for a system of nonlinear integral equation and on operator equation is given here.

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2. Preliminaries:

A Banach algebra $\mathbb{A}$ is said to be a C^*-algebra with the involution map $* : \mathbb{A} \to \mathbb{A}$ satisfying the following conditions: ([11],[21])

(i) $a^{**} = a, \ a \in \mathbb{A}$
(ii) $(ab)^* = b^* a^*, \ a, b \in \mathbb{A}$
(iii) $(\alpha a + \mu b)^* = \bar{\alpha} a^* + \bar{\mu} b^*, \ \alpha, \mu \in \mathbb{C}$
(iv) $\|a^* a\| = \|a\|^2$, (which easily shows that $\|a^*\| = \|a\|$)

Let $\mathbb{A}$ be an unital C^*-algebra with the identity $1_{\mathbb{A}}$ and θ be the zero element of $\mathbb{A}$. Every element of the set $\mathbb{A}_+ = \{x \in \mathbb{A} : x = x^*\}$ is called positive (written as $x \succeq \theta$) and a spectrum $\sigma(x) \subset \mathbb{R}_+$, where $\sigma(x) = \{\alpha \in \mathbb{R} : |\alpha 1_{\mathbb{A}} - x| = 0\}$.

A partial ordering " $\succeq$ " on $\mathbb{A}$, behaves as $x \succeq y \Leftrightarrow y - x \in \mathbb{A}_+$. For every element $\theta \preceq x \in \mathbb{A}$ has a unique positive square root, i.e., $|x| = (x^* x)^{\frac{1}{2}}$.

2.1 Lemma: [11] Suppose that $\mathbb{A}$ is a unital C^*-algebra with a unit $1_{\mathbb{A}}$.

(i) for any $x \in \mathbb{A}_+$, $x \preceq 1_{\mathbb{A}}$ if and only if $x \leqslant 1$,
(ii) if $x \in \mathbb{A}_+$ and $\|a\| < \frac{1}{2}$, then $1_{\mathbb{A}} - x$ is invertable and $\|x(1_{\mathbb{A}} - x)^{-1}\| < 1$
(iii) let $x, y \in \mathbb{A}_+$ with $x, y \succeq \theta$ and $xy = yx$ then $xy \succeq \theta$,
(iii) let $x \in \mathbb{A}' = \{x \in \mathbb{A} : xy = yx, y \in \mathbb{A}\}$, if $y, z \in \mathbb{A}$ and $y \succeq z \succeq \theta$, and $1_{\mathbb{A}} - x \in \mathbb{A}'_+$ is an invertable operator , then $y(1_{\mathbb{A}} - x)^{-1} \succeq z(1_{\mathbb{A}} - x)^{-1}$

In C^*-algebra, $x, y \succeq \theta \not\Rightarrow xy \succeq \theta$ in general.

2.2 Definition: [18] Let Z be a non empty set, and $K \in A'$ with $K \succeq 1_{\mathbb{A}}$. A mapping $d : Z \times Z \to \mathbb{A}^+$ is called a C^*- algebra valued b-metric on Z, if it satisfies the following three conditions :

(i) for all $a, b \in Z, d(a, b) \succeq \theta$ and $d(a, b) = \theta$ if and only if $a = b$;
(ii) $d(a, b) = d(b, a)$, for all $a, b \in Z$
(iii) $d(a, b) \preceq K[d(a, c) + d(c, b)]$, for all $a, b, c \in Z$;

and $(Z, \mathbb{A}, d)$ is said to be C^*- algebra valued b-metric space.

2.3 Definition: [13] Let Z be a non empty set and $K \geqslant 1$ be a real number. A mapping $\Omega : (0, \infty) \times Z \times Z \to [0, \infty]$ is called a modular b-metric on Z, if it satisfies the following three conditions :

(i) for $a, b \in Z, \Omega_\alpha(a, b) = 0$ for all $\alpha > 0$ if and only if $a = b$;
(ii) $\Omega_\alpha(a, b) = \Omega_\alpha(b, a)$ for all $\alpha > 0$ and $a, b \in Z$;
(iii) $\Omega_{\alpha+\mu}(a, b) \leqslant K[\Omega_\alpha(a, c) + \Omega_\mu(c, b)]$ for all $\alpha > 0 \, \mu > 0$ and $a, b, c \in Z$;

and (Z, Ω) is said to be modular b-metric space.

In the following the concept of C^*- algebra valued modular b-metric space is introduced.

2.4 Definition: Let Z be a non empty set, and $K \in A_+$ with $K \succeq 1_{\mathbb{A}}$. A mapping $\Omega : (0, \infty) \times Z \times Z \to \mathbb{A}^+$ is called a C^*- algebra valued b-modular metric (or C^* m.b-m) on Z if it satisfies the following three conditions :

(i) for $a, b \in Z, \Omega_\alpha(a, b) = \theta$ for all $\alpha > 0$ if and only if $a = b$;
(ii) $\Omega_\alpha(a, b) = \Omega_\alpha(b, a)$ for all $\alpha > 0$ and $a, b \in Z$;
(iii) $\Omega_{\alpha+\mu}(a, b) \preceq K[\Omega_\alpha(a, c) + \Omega_\mu(c, b)]$ for all $\alpha > 0 \, \mu > 0$ and $a, b, c \in Z$;

and $(Z, \mathbb{A}, \Omega)$ is said to be C^*- algebra valued modular b-metric space.

Now we discuss some properties and definitions on C^*- algebra valued modular b-metric spaces as follows:

(a) If we replace the above condition (i) by $(i^/)$ for $a \in Z, \Omega_\alpha(a,a) = \theta$ for all $\alpha > 0$, then Ω is said to be a C^*- algebra valued pseudo modular b-metric on Z satisfying with (ii) and (iii).

(b) If Ω satisfies $(i^/)$, $(ii^/)$ for $a, b \in Z$, if there exists a number $\alpha > 0$, possibly depending on a and b, such that $\Omega_\alpha(a,b) = \theta$, then $a = b$ and (iii) then Ω is called a C^*- algebra valued strict modular b-metric on Z.

(c) The essential property on a set Z of a C^*- algebra valued modular b-metric Ω is that for any $a, b \in Z$ the function $0 < \alpha \to \Omega_\alpha(a,b) \in \mathbb{A}$ is non increasing on $(0, \infty)$. In fact, if $0 < \mu < \alpha$, then we have

$$\Omega_\alpha(a,b) \preceq K[\Omega_{\alpha-\mu}(a,a) + \Omega_\mu(a,b)] = K\Omega_\mu(a,b).\ (K \geqslant 1).$$

(d) It follows there exists right limit $\Omega_{\alpha+0}(a,b) := \lim_{\epsilon\to+0} \Omega_{\alpha+\epsilon}(a,b)$ and left limit $\Omega_{\alpha-0}(a,b) := \lim_{\epsilon\to+0} \Omega_{\alpha-\epsilon}(a,b)$ in $\mathbb{A}$ for each point $\alpha > 0$ such that:

$$\Omega_{\alpha+0}(a,b) \preceq K\Omega_\alpha(a,b)\ and\ \Omega_\alpha(a,b) \preceq K\Omega_{\alpha-0}(a,b),$$

$$i.e.,\ \frac{1}{K}\Omega_{\alpha+0}(a,b) \preceq \Omega_\alpha(a,b) \preceq K\Omega_{\alpha-0}(a,b).$$

(e) It can be easily checked as in [5] that, if $a_0 \in Z$ the set

$$Z_\Omega = \{a \in Z : \lim_{\alpha\to\infty} \Omega_\alpha(a, a_0) = \theta\},$$

is a C^*- algebra valued b-metric space with the metric $d^0_\Omega : Z_\Omega \times Z_\Omega \to \mathbb{A}$ which is given by

$$d^0_\Omega(a,b) = \inf\{\alpha > 0 : \|\Omega_\alpha(a,b)\| \leqslant \alpha\} \text{ for all } a, b \in Z_\Omega,$$

called C^*- algebra valued modular b-metric space.

(f) A C^*- algebra valued modular b-metric Ω on Z is said to be convex if (iii) is replaced by the following condition for $a, b, c \in Z$:

$$\Omega_{\alpha+\mu}(a,b) \preceq K[\frac{\alpha}{K(\alpha+\mu)}\Omega_\alpha(a,c) + \frac{\mu}{K(\alpha+\mu)}\Omega_\mu(c,b)] \text{ for all } \alpha > 0\, \mu > 0$$

If Ω is convex on Z then Z_Ω is equal to

$$Z^*_\Omega = \{a \in Z : \exists\alpha = \alpha(x) > 0\ such\ that\ \|\Omega_\alpha(a, a_0)\| < \infty\},$$

equipped with the metric $d^*_\Omega : Z^*_\Omega \times Z^*_\Omega \to \mathbb{A}$ is given by

$$d^*_\Omega = \inf\{\alpha > 0 : \|\Omega_\alpha(a,b)\| \leq 1\} \text{ for all } a, b \in Z_\Omega.$$

(g) It is easy to see that if Z is a real linear space, $\rho : Z \to \mathbb{A}$ and

$$\Omega_\alpha(a,b) = \rho(\frac{a-b}{\alpha}) \text{ for all } \alpha > 0 \text{ and } a, b \in Z,$$

then ρ is C^*- algebra- valued modular on X, and Ω satisfy the following two conditions :

(i) $\Omega_\alpha(\mu a, 0) = \Omega_{\frac{\alpha}{\mu}}(a, 0)$ for all $\alpha,\ \mu > 0$ and $a \in Z$;

(ii) $\Omega_\alpha(a + c, b + c) = \Omega_\alpha(a,b)$ for all $\alpha > 0$ and $a, b, c \in Z$.

2.5 Definition: Let Z_Ω be a C^*- algebra valued modular b-metric space for each $\alpha > 0$

1) A sequence $\{a_n\}_{n\in\mathbb{N}}$ in Z_Ω is said to be Ω- b-convergent to $a \in Z_\Omega$ with respect to $\mathbb{A}$,
if $\lim_{n\to\infty} \Omega_\alpha(a_n, a) = \theta$.
2) A sequence $\{a_n\}_{n\in\mathbb{N}}$ in Z_Ω is said to be Ω- b-Cauchy with respect to $\mathbb{A}$,
if $\lim_{(m,n)\to\infty} \Omega_\alpha(a_m, a_n) = \theta$.
3) A mapping T is said to be Ω-b-continuous with respect to $\mathbb{A}$ in $B \subseteq Z_\Omega$ if for every sequence $\{a_n\}n \in N \subseteq B$ such that $\lim_{n\to\infty} \Omega_\alpha(a_n, z) = \theta$, then $\lim_{n\to\infty} \Omega_\alpha(Ta_n, Tz) = \theta$.
4) $B \subseteq Z_\Omega$, is said to be Ω- closed to $a \in Z_\Omega$ with respect to $\mathbb{A}$ if the limit of the Ω- b-convergent sequence of B always belong to B.
5) Z_Ω is said to be Ω-complete if any Ω- b-Cauchy sequence with respect to $\mathbb{A}$ is Ω-b-convergent.
6) A subset B of Z_Ω is said to be Ω- b-bounded with respect to $\mathbb{A}$ if for each $\alpha > 0$

$$\delta_\Omega(B) = sup\{\|\Omega_\alpha(a, b)\|; a, b \in B\} < \infty,$$

where, $\delta_\Omega(B)$ denotes the diameter of B in the C^*- algebra valued modular b-metric space.

2.6 Example Let $Z = \mathbb{R}$ and consider, $\mathbb{A} = M_2(\mathbb{R})$. Define a norm on $\mathbb{A}$ by $\|C\| = \left(\sum_{i,j=1}^{2} |c_{ij}|^2\right)^{\frac{1}{2}}$ and a convolution mapping $* : \mathbb{A} \to \mathbb{A}$ with $C^* = C$ for all $A \in \mathbb{A}$. Clearly, $\mathbb{A}$ is a C^*-algebra. For

$$C = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix},\ D = \begin{pmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{pmatrix} \in \mathbb{A} = M_2(\mathbb{R})$$

denote $D \succeq C \Leftrightarrow (d_{ij} - c_{ij}) \geqslant 0$ for all $i, j = 1, 2$.
Define $\Omega : (0, \infty) \times Z \times Z \to \mathbb{A}$ by $\Omega_\alpha(a, b) = diag(\beta \left|\frac{a-b}{\alpha}\right|^2, \beta \left|\frac{a-b}{\alpha}\right|^2)$, where $\beta > 0$.
By the inequality $\left|\frac{a-b}{\alpha+\mu}\right|^2 \leqslant 2^p[\left|\frac{a-c}{\alpha}\right|^2 + \left|\frac{c-b}{\mu}\right|^2]$; $(2^p > 1)$, we can get

$$diag(\beta\left|\frac{a-b}{\alpha+\mu}\right|^2, \beta\left|\frac{a-b}{\alpha+\mu}\right|^2) \preceq 2^p[diag(\beta\left|\frac{a-c}{\alpha}\right|^2, \beta\left|\frac{a-c}{\alpha}\right|^2) + diag(\beta\left|\frac{c-b}{\alpha}\right|^2, \beta\left|\frac{c-b}{\alpha}\right|^2)]$$

For $a \succ c \succ b$ the inequality $\left|\frac{a-b}{\alpha+\mu}\right|^2 \leqslant \left|\frac{a-c}{\alpha}\right|^2 + \left|\frac{c-b}{\mu}\right|^2$ is impossible.
It is easy to check that Ω satisfies all the conditions of C^*-algebra valued modular b-metric space. But Z_Ω cannot be a C^*-algebra valued modular metric space.

2.7 Example Let $Z = L^\infty(E)$ and $H = L^2(E)$, E be a Lebesgue measurable set, and $\mathbb{A} = B(H)$, the set of bounded linear operator on Hilbert space H. Define $\Omega : (0, \infty) \times Z \times Z \to B(H)_+$ by

$$\Omega_\alpha(f, g) = \pi_{\left|\frac{f-g}{\alpha}\right|^p} \text{ for all } f, g \in Z,\ \alpha > 0, p \geqslant 1,$$

where $\pi_h : H \to H$ is the multiplication operator defined by $\pi_h(\phi) = h.\phi$, $\phi \in H$. Then $(Z_\Omega, B(H), \Omega)$ is a Ω-b-complete C^* modular b-metric space with $\|K\| \geqslant 2^p$.

It suffices to verify the completeness of Z_Ω . For this, let $\{f_n\}$ be a Ω-b-Cauchy sequence with respect to $B(H)$, that is for an arbitrary $\epsilon > 0$, there is $N \in \mathbb{N}$ such that for all $m, n \geqslant N$,

$$\|\Omega_\alpha(f_m, f_n)\| = \|\pi_{|\frac{f_m - f_n}{\alpha}|^p}\| = \left\| \left| \frac{f_m - f_n}{\alpha} \right|^p \right\|_\infty \leqslant \epsilon$$

so $\{f_n\}$ is a Cauchy sequence in Banach space Z. Hence, there is a function $f \in X$ and $N_1 \in \mathbb{N}$ such that $\||\frac{f_m - f}{\alpha}|^p\|_\infty \leqslant \epsilon$. It implies that

$$\|\Omega_\alpha(f_m, f)\| = \|\pi_{|\frac{f_m - f}{\alpha}|^p}\| = \left\| \left| \frac{f_m - f}{\alpha} \right|^p \right\|_\infty \leqslant \epsilon$$

Consequently, the sequence $\{f_n\}$ is a Ω-b-convergent sequence in Z_Ω and so Z_Ω is a Ω-b-complete C^* m.b-m. space.

2.8 Example: Let $Z = l^\infty(S)$ and $H = L^2(S)$, S be a non empty set, and $\mathbb{A} = B(H)$. Define $\Omega : (0, \infty) \times Z \times Z \to B(H)_+$ by

$$\Omega_\alpha(\{f_n\}, \{g_n\}) = \pi_{\left|\frac{\{f_n\} - \{g_n\}}{\alpha}\right|^p} \text{ for all } \{f_n\}, \{g_n\} \in Z,\ \alpha > 0, p \geqslant 1,$$

where $\pi_h : H \to H$ is the multiplication operator defined by $\pi_h(\phi) = h.\phi$, $\phi \in H$. Then $(Z_\Omega, B(H), \Omega)$ is a Ω-b-complete C^* modular b-metric space with $\|K\| \geqslant 2^p$.

2.9 Definition: Let Z_Ω be a C^*- algebra valued modular b-metric space. Let T_1, T_2 be two self mappings of Z_Ω, a point a in Z_Ω is called a coincidence point of T_1 and T_2 if and only if $T_1 a = T_2 a$. We shall call $q = T_1 a = T_2 a$ a point of coincidence of T_1 and T_2. Moreover, T_1 and T_2 is said to be weakly compatible if they commute at coincidence points.

Let $C(T_1, T_2)$ denote the set of coincidence points and $PC(T_1, T_2)$ denote points of coincidence of the pair (T_1, T_2), respectively.

2.10 Definition: Let Z_Ω be a C^*- algebra valued modular b-metric space and $T_1, T_2 : Z_\Omega \to Z_\Omega$ be two mappings. Then (T_1, T_2) is called a $\mathcal{JHR}$-operator pair if $PC(T_1, T_2) \neq \phi$ and there exists a sequence $\{a_n\}$ in Z_Ω such that $\lim_{n\to\infty} T_1 a_n = \lim_{n\to\infty} T_2 a_n = \beta \in Z_\Omega$ and which satisfies

$$\lim_{n\to\infty} \|\Omega_\alpha(a_n, \beta)\| \leq \delta_\Omega(PC(T_1, T_2)), \text{ for all } \alpha > 0.$$

2.11 Example: Let $(Z, \mathbb{A}, \Omega)$ as in Example 2.6 and define self mappings $T_1, T_2 : Z_\Omega \to Z_\Omega$ such that $T_1 = x^2$ and $T_2 = x^3$. So, $C(T_1, T_2) = \{1\}$, $PC(T_1, T_2) = \{1\}$ and $\delta_\Omega(PC(T_1, T_2)) = 0$. Define a sequence $\{x_n\} = \frac{n}{n+1}$, $n = 1, 2, ...$ in Z_Ω. Then $\lim_{n\to\infty} T_1 x_n = q = 1 = \lim_{n\to\infty} T_2 x_n$, and for all $\alpha > 0$,

$$0 = \lim_{n\to\infty} \|\Omega_\alpha(x_n, q)\| \leqslant \delta_\Omega(PC(T_1, T_2)) = 0.$$

Hence (T_1, T_2) is a $\mathcal{JHR}$ operator pair.

It can be examined that all the properties of $\mathcal{JHR}$ operator pairs as discussed in [21] on C^*-alebra valued modular metric spaces are also same in C^*-algebra valued modular b-metric spaces but converse not necessarily true.

2.12 Definition: [21] Suppose $\mathbb{A}$ be a C^*-algebra, then a continuous function $H : \mathbb{A}^+ \times \mathbb{A}^+ \to \mathbb{A}$ is called C_*-class function if for any $A, B \in \mathbb{A}^+$, the following conditions hold:

i) $H(A, B) \preceq A$;
ii) $H(A, B) = A$ implies that either $A = \theta$ or $B = \theta$.

2.13 Definition: [21] A tripled (ψ, φ, H_*) where $\psi : \mathbb{A}^+ \to \mathbb{A}^+$ in Ψ(set of all continuous functions), $\varphi : \mathbb{A}^+ \to \mathbb{A}^+$ in Φ_v (the class of functions) and $H_* : \mathbb{A}^+ \times \mathbb{A}^+ \to \mathbb{A}^+$ in C_* is said to be monotone if for any $A, B \in \mathbb{A}^+$

$$A \preceq B \Rightarrow H_*(\psi(A), \varphi(A)) \preceq H_*(\psi(B), \varphi(B)),$$

and (ψ, φ, H_*) is said to be strictly monotone if for any $A, B \in \mathbb{A}^+$

$$A \prec B \Rightarrow F_*(\psi(A), \varphi(A)) \prec F_*(\psi(B), \varphi(B)).$$

2.14 Lemma: Let $\{a_m\}_{m\in\mathbb{N}}$ and $\{b_n\}_{n\in\mathbb{N}}$ be two positive integer sequences $(n > m)$. Then in modular metric spaces and modular b-metric spaces the following inequalities holds for each $\alpha > 0$ and $d \in \mathbb{N}$:

(i) $\Omega_\alpha(a_m, b_n) \leqslant \Omega_{\frac{\alpha}{d}}(a_m, b_{m+1}) + \Omega_{\frac{\alpha}{d}}(b_{m+1}, b_{m+2}) + ... + \Omega_{\frac{\alpha}{d}}(b_{n-1}, b_n)$

(ii) $\Omega_\alpha(a_m, b_n) \leqslant K\Omega_{\frac{\alpha}{d}}(a_m, b_{m+1}) + K^2\Omega_{\frac{\alpha}{d}}(b_{m+1}, b_{m+2}) + ... + K^d\Omega_{\frac{\alpha}{d}}(b_{n-1}, b_n)$

Proof. For $K = 1$, (i) holds automatically as follows.
(ii)

$$\begin{aligned}\Omega_\alpha(a_m, b_n) = \Omega_{\frac{\alpha+(d-1)\alpha}{d}}(a_m, b_n) &\leqslant K[\Omega_{\frac{\alpha}{d}}(a_m, b_{m+1}) + \Omega_{\frac{(d-1)\alpha}{d}}(b_{m+1}, b_n)] \\ &\leqslant K\Omega_{\frac{\alpha}{d}}(a_m, b_{m+1}) + K^2[\Omega_{\frac{\alpha}{d}}(b_{m+1}, b_{m+2}) + \Omega_{\frac{(d-2)\alpha}{d}}(b_{m+2}, b_n)] \\ &\quad ... \\ &\leqslant K\Omega_{\frac{\alpha}{d}}(a_m, b_{m+1}) + K^2\Omega_{\frac{\alpha}{d}}(b_{m+1}, b_{m+2}) + ... + K^d\Omega_{\frac{\alpha}{d}}(b_{n-1}, b_n).\end{aligned}$$

□

2.15 Lemma: Let $\{y_n\}$ be a Ω-b-Cauchy sequence in Z_Ω with $K \geqslant 1$. If $\{y_n\}$ is not a Ω-b-Cauchy sequence in Z_Ω, then there exists $\alpha_0 > 0$ and $\epsilon_0 > 0$, and two sequences $\{a_m\}_{m\in\mathbb{N}}$ and $\{b_n\}_{n\in\mathbb{N}}$ of positive integers such that

(i) $m_i > n_i + 1$ and $n_i \to \infty$ as $i \to \infty$,

(ii) $\epsilon_0 \leqslant \|\Omega_{\alpha_0}(y_{n_i}, y_{m_i})\|$ and $\|\Omega_{\alpha_0}(y_{n_i}, y_{n_i+1})\| \leqslant \frac{\epsilon_0}{K} < \epsilon_0$.

3. Main Results

3.1 Theorem: Let $(Z_\Omega, \mathbb{A}, \Omega)$ be a Ω-b-complete C^*- algebra valued modular b-metric space with $1_\mathbb{A} \preceq K$. Let A, B, C and D be four self-mappings on Z_Ω, and satisfies the following conditions, for each $a, b \in Z_\Omega$ and $\alpha > 0$:

(i) $A(Z_\Omega) \subseteq C(Z_\Omega)$ and $B(Z_\Omega) \subseteq D(Z_\Omega)$,

(ii)

$$\psi(\|\Omega_\alpha(Aa, Bb)\|1_\mathbb{A}) \preceq F_*(\psi(M(a, b)), \phi(M(a, b)),$$

for which, $\psi \in \Psi$, $\phi \in \Phi_u$ and $F_* \in C_*$ such that (ψ, ϕ, F_*) is strictly monotone and

$$M(a, b) = \|j\|^2\|\Omega_\alpha(Da, Cb)\|1_\mathbb{A} + \|L\|^2 min\{\|\Omega_\alpha(Aa, Cb)\|, \|\Omega_\alpha(Bb, Da)\|, \|\Omega_\alpha(Aa, Da)\|\}1_\mathbb{A}$$

where $j, L \in \mathbb{A}$ with $0 < \|j\| < 1$, $\|L\| > 0$ and $\|\Omega_\alpha(Aa, Bb)\| < \infty$.

Moreover , if D is Ω-b-continuous and the pairs $(A, D), (B, C)$ are $\mathcal{JHR}$-operator pair and weakly compatible, respectively, then A, B, C and D have a unique common fixed point in Z_Ω.

Proof. Let A, B, C and D be four self-mappings on Z_Ω such that $A(Z_\Omega) \subseteq C(Z_\Omega)$ and $B(Z_\Omega) \subseteq D(Z_\Omega)$. For any $a_0 \in Z_\Omega$, there exists a point a_1 such that $Ca_1 = Aa_0$, and for this a_1, there exists a point $a_2 \in Z_\Omega$ such that $Da_2 = Ba_1$. Thus inductively, for a sequence $\{y_n\}$ in Z_Ω we have,

$$y_{2n} = Ca_{2n+1} = Aa_{2n} \text{ and } y_{2n+1} = Da_{2n+2} = Ba_{2n+1},\ n = 1, 2, 3, ...$$

$$\begin{aligned}
\psi(\|\Omega_\alpha(y_{2n}, y_{2n+1})\|1_{\mathbb{A}}) &= \psi(\|\Omega_\alpha(Aa_{2n}, Ba_{2n+1})\|1_{\mathbb{A}}) \\
&\preceq F_*(\psi(M(a_{2n}, a_{2n+1})), \varphi(M(a_{2n}, a_{2n+1}))) \\
&\preceq \psi(M(a_{2n}, a_{2n+1})) \\
&= \psi(\|j\|^2\|\Omega_\alpha(Da_{2n}, Ca_{2n+1})\|1_{\mathbb{A}} + \|L\|^2 min\{\|\Omega_\alpha(Aa_{2n}, Ca_{2n+1})\|, \\
&\quad \|\Omega_\alpha(Ba_{2n+1}, Da_{2n})\|, \|\Omega_\alpha(Aa_{2n}, Da_{2n})\|\}1_{\mathbb{A}}) \\
\Rightarrow \psi(\|\Omega_\alpha(y_{2n}, y_{2n+1})\|1_{\mathbb{A}}) &\preceq \psi(\|j\|^2\|\Omega_\alpha(Da_{2n}, Ca_{2n+1})1_{\mathbb{A}}) \\
&\preceq \psi(\|j\|^2\|\Omega_\alpha(y_{2n-1}, y_{2n})\|1_{\mathbb{A}})
\end{aligned}$$

If $\Omega_\alpha(y_{2n}, y_{2n+1}) \succ \Omega_\alpha(y_{2n-1}, y_{2n})$, then

$$\psi(\|\Omega_\alpha(y_{2n}, y_{2n+1})\|1_{\mathbb{A}}) \prec \psi(\|\Omega_\alpha(y_{2n+1}, y_{2n})\|1_{\mathbb{A}})$$

which is a contraction. Therefore,

$$\text{(1)} \qquad \Omega_\alpha(y_{2n}, y_{2n+1}) \preceq \Omega_\alpha(y_{2n-1}, y_{2n})$$

Similarly,

$$\begin{aligned}
\psi(\|\Omega_\alpha(y_{2n+2}, y_{2n+1})\|1_{\mathbb{A}}) &= \psi(\|\Omega_\alpha(Aa_{2n+2}, Ba_{2n})\|1_{\mathbb{A}}) \\
&\preceq F_*(\psi(M(a_{2n+2}, a_{2n+1})), \varphi(M(a_{2n+2}, a_{2n+1}))) \\
&\preceq \psi(M(a_{2n+2}, a_{2n+1})) \\
&= \psi(\|j\|^2\|\Omega_\alpha(Da_{2n+2}, Ca_{2n+1})\|1_{\mathbb{A}} + \|L\|^2 min\{\|\Omega_\alpha(Aa_{2n+2}, Ca_{2n+1} \\
&\quad \|\Omega_\alpha(Ba_{2n+1}, Da_{2n+2})\|, \|\Omega_\alpha(Aa_{2n+2}, Da_{2n+2})\|\}1_{\mathbb{A}}) \\
\Rightarrow \psi(\|\Omega_\alpha(y_{2n+2}, y_{2n+1})\|1_{\mathbb{A}}) &\prec \psi(\|\Omega_\alpha(y_{2n+1}, y_{2n})\|1_{\mathbb{A}})
\end{aligned}$$

If $\Omega_\alpha(y_{2n+2}, y_{2n+1}) \succ \Omega_\alpha(y_{2n+1}, y_{2n})$, then

$$\psi(\|\Omega_\alpha(y_{2n+2}, y_{2n+1})\|1_{\mathbb{A}}) \prec \psi(\|\Omega_\alpha(y_{2n+2}, y_{2n+1})\|1_{\mathbb{A}})$$

which is a contraction. Therefore,

$$\text{(2)} \qquad \Omega_\alpha(y_{2n+2}, y_{2n+1}) \preceq \Omega_\alpha(y_{2n+1}, y_{2n})$$

From equation (1) and (2), we can conclude that $\Omega_\alpha(y_n, y_{n+1}) \preceq \Omega_\alpha(y_{n-1}, y_n)$.
So, $\{\|\Omega_\alpha(y_n, y_{n+1})\|1_{\mathbb{A}}\}_{n\in\mathbb{N}}$ is a non increasing sequence in $\mathbb{A}_+$.
Again we have,

$$\begin{aligned}
\psi(\|\Omega_\alpha(y_2, y_1)\|1_{\mathbb{A}}) &= \psi(\|\Omega_\alpha(Aa_2, Ba_1)\|1_{\mathbb{A}}) \\
&\preceq F_*(\psi(M(a_2, a_1)), \varphi(M(a_2, a_1))) \\
&\preceq \psi(M(a_2, a_1)) \\
&= \psi(\|j\|^2\|\Omega_\alpha(Da_2, Ca_1)\|1_{\mathbb{A}} + \|L\|^2 min\{\|\Omega_\alpha(Aa_2, Ca_1)\|, \\
&\quad \|\Omega_\alpha(Ba_1, Da_2)\|, \|\Omega_\alpha(Aa_2, Da_2)\|\}1_{\mathbb{A}}) \\
\Rightarrow \psi(\|\Omega_\alpha(y_2, y_1)\|1_{\mathbb{A}}) &\preceq \psi(\|j\|^2\|\Omega_\alpha(y_1, y_0)\|1_{\mathbb{A}})
\end{aligned}$$

and

$$\begin{aligned}\psi(\|\Omega_\alpha(y_3,y_2)\|1_{\mathbb{A}}) &= \psi(\|\Omega_\alpha(Aa_2,Ba_3)\|1_{\mathbb{A}})\\ &\preceq F_*(\psi(M(a_2,a_3)),\varphi(M(a_2,a_3)))\\ &\preceq \psi(M(a_2,a_3))\\ &= \psi(\|j\|^2\|\Omega_\alpha(Da_2,Ca_3)\|1_{\mathbb{A}}+\|L\|^2 min\{\|\Omega_\alpha(Aa_2,Ca_3)\|,\\ &\qquad \|\Omega_\alpha(Ba_3,Da_2)\|,\|\Omega_\alpha(Aa_2,Da_2)\|\}1_{\mathbb{A}})\end{aligned}$$

$$\Rightarrow \psi(\|\Omega_\alpha(y_3,y_2)\|1_{\mathbb{A}}) \preceq \psi(\|j\|^2\|\Omega_\alpha(y_2,y_1)\|1_{\mathbb{A}})$$

In general,

$$\|\Omega_\alpha(y_{n+1},y_n)\|1_{\mathbb{A}} \preceq \|j\|^{2n}\|\Omega_\alpha(y_1,y_0)\|1_{\mathbb{A}} \tag{3}$$

$$\Rightarrow \lim_{n\to\infty}\|\Omega_\alpha(y_{n+1},y_n)\|1_{\mathbb{A}} \preceq \lim_{n\to\infty}\|j\|^{2n}\|\Omega_\alpha(y_1,y_0)\|1_{\mathbb{A}}=\theta.$$

$$So,\ \lim_{n\to\infty}\Omega_\alpha(y_{n+1},y_n)=\theta, \text{for all } \alpha>0 \tag{4}$$

Now, by using equation (3) and (4), we will show that $\{y_n\}$ is a Ω-b-Cauchy Sequence. From the Lemma 2.15, if $\{y_n\}$ is not a Ω-b-Cauchy Sequence then there exists $\epsilon_0>0$, $\alpha_0>0$ and two positive sequences $\{m_i\}$and $\{n_i\}$ such that

$A)$ $m_i > n_i+1$

$B)$ $\|\Omega_{\alpha_0}(y_{m_i},y_{n_i})\| \geqslant \epsilon_0$ *and* $\|\Omega_{\alpha_0}(y_{m_i-1},y_{n_i})\| \leqslant \frac{\epsilon_0}{K} < \epsilon_0$

$$\begin{aligned}\epsilon_0 \leqslant \|\Omega_{\alpha_0}(y_{n_i},y_{m_i})\| &\leqslant K[\|\Omega_{\frac{\alpha_0}{2}}(y_{n_i},y_{m_i-1})\|+\|\Omega_{\frac{\alpha_0}{2}}(y_{m_i-1},y_{m_i})\|]\\ &\leqslant K[\frac{\epsilon_0}{K}+\|\Omega_{\frac{\alpha_0}{2}}(y_{m_i-1},y_{m_i})\|\end{aligned}$$

Let $i\to\infty$, and by (4), we get $\lim_{i\to\infty}\|\Omega_{\alpha_0}(y_{n_i},y_{m_i})\|=\epsilon_0$.
Again,

$$\epsilon_0 \leqslant \|\Omega_{\alpha_0}(y_{n_i},y_{m_i})\| \leqslant K[\|\Omega_{\frac{\alpha_0}{2}}(y_{n_i},y_{n_i+1})\|+\|\Omega_{\frac{\alpha_0}{2}}(y_{n_i+1},y_{m_i})\|] \tag{5}$$

Assume both m_i and n_i are even. Now,

$$\begin{aligned}\psi(\|\Omega_{\frac{\alpha_0}{2}}(y_{m_i},y_{n_i+1})\|1_{\mathbb{A}}) &= \psi(\|\Omega_{\frac{\alpha_0}{2}}(Aa_{m_i},Ba_{n_i+1})\|1_{\mathbb{A}})\\ &\preceq F_*(\psi(M(a_{m_i},a_{n_i+1})),\varphi(M(a_{m_i},a_{n_i+1})))\\ &\preceq \psi(M(a_{m_i},a_{n_i+1}))\\ &= \psi(\|j\|^2\|\Omega_{\frac{\alpha_0}{2}}(Da_{m_i},Ca_{n_i+1})\|1_{\mathbb{A}}+\|L\|^2 min\{\|\Omega_{\frac{\alpha_0}{2}}(Aa_{m_i},Ca_{n_i+1})\|,\\ &\qquad \|\Omega_{\frac{\alpha_0}{2}}(Ba_{n_i+1},Da_{m_i})\|,\|\Omega_{\frac{\alpha_0}{2}}(Aa_{m_i},Da_{m_i})\|\}1_{\mathbb{A}})\end{aligned}$$

$$\begin{aligned}\Rightarrow \psi(\|\Omega_{\frac{\alpha_0}{2}}(y_{m_i},y_{n_i+1})\|1_{\mathbb{A}}) &\preceq \psi(\|j\|^2\|\Omega_{\frac{\alpha_0}{2}}(y_{m_i-1},y_{n_i})\|1_{\mathbb{A}}+\|L\|^2 min\{\|\Omega_{\frac{\alpha_0}{2}}(y_{m_i},y_{n_i})\|,)\\ &\qquad \|\Omega_{\frac{\alpha_0}{2}}(y_{n_i+1},y_{m_i-1})\|,\|\Omega_{\frac{\alpha_0}{2}}(y_{m_i},y_{m_i-1})\|\}1_{\mathbb{A}})\\ &\preceq \psi(\|j\|^2\|\Omega_{\frac{\alpha_0}{2}}(y_{m_i-1},y_{n_i})\|1_{\mathbb{A}})\end{aligned}$$

Let $i\to\infty$, and using equation (4), (5), (B), we have

$$\psi(\epsilon_0 1_{\mathbb{A}}) \preceq \psi(\|j\|^2\epsilon_0 1_{\mathbb{A}})$$
$$\psi(\epsilon_0 1_{\mathbb{A}}) \prec \psi(\epsilon_0 1_{\mathbb{A}})$$

which is a contraction. So, $\{y_n\}$ is a Ω-b-Cauchy sequence.
By hypothesis $PC(A,D) \neq \phi$, there exists a point $q \in Z_\Omega$ such that $Ap = Dp = q$. Since, $\{y_n\}$ is a Ω-b-Cauchy sequence in Z_Ω and Z_Ω is complete, so there exists a point $r \in Z_\Omega$ such that for all $\alpha > 0$ $\lim_{n\to\infty} \Omega_\alpha(y_n, r) = \theta$, and for all $\alpha > 0$

$$\lim_{n\to\infty} \Omega_\alpha(Aa_{2n}, r) = \lim_{n\to\infty} \Omega_\alpha(Ca_{2n+1}, r) = \lim_{n\to\infty} \Omega_\alpha(Da_{2n}, r) = \lim_{n\to\infty} \Omega_\alpha(Ba_{2n+1}, r) = \theta$$

$$(6) \quad \psi(\|\Omega_\alpha(Ap, Ba_{2n+1})\|1_{\mathbb{A}}) \preceq F_*(\psi(M(p, a_{2n+1})), \varphi(M(p, a_{2n+1})))$$

where,

$$\begin{aligned} M(p, a_{2n+1}) = & \|j\|^2\|\Omega_\alpha(Dp, Ca_{2n+1})\|1_{\mathbb{A}} + \|L\|^2 min\{\|\Omega_\alpha(Ap, Ca_{2n+1})\|, \\ & \|\Omega_\alpha(Ba_{2n+1}, Dp)\|, \|\Omega_\alpha(Ap, Dp)\|\}1_{\mathbb{A}} \\ = & \|j\|^2\|\Omega_\alpha(Dp, Ca_{2n+1})\|1_{\mathbb{A}} \end{aligned}$$

Let $n \to \infty$, equation (6) equals

$$\begin{aligned} \lim_{n\to\infty} \psi(\|\Omega_\alpha(Ap, Ba_{2n+1})\|1_{\mathbb{A}}) \preceq & \lim_{n\to\infty} F_*(\psi(\|j\|^2\|\Omega_\alpha(Dp, Ca_{2n+1})\|1_{\mathbb{A}}), \\ & \varphi(\|j\|^2\|\Omega_\alpha(Dp, Ca_{2n+1})\|1_{\mathbb{A}})) \\ \Rightarrow \psi(\|\Omega_\alpha(q, r)\|1_{\mathbb{A}}) \preceq & F_*(\psi(\|j\|^2\|\Omega_\alpha(q, r)\|1_{\mathbb{A}}), \varphi(\|j\|^2\|\Omega_\alpha(q, r)\|1_{\mathbb{A}})) \\ \preceq & F_*(\psi(\|\Omega_\alpha(q, r)\|1_{\mathbb{A}}), \varphi(\|\Omega_\alpha(q, r)\|1_{\mathbb{A}})) \\ \preceq & \psi(\|\Omega_\alpha(q, r)\|1_{\mathbb{A}}) \end{aligned}$$

So, by the definition of C_*-class function we get $\psi(\|\Omega_\alpha(q, r)\|1_{\mathbb{A}}) = \theta$ or $\varphi(\|\Omega_\alpha(q, r)\|1_{\mathbb{A}}) = \theta$, which implies $q = r$. Hence $Ap = Dp = r$. If there exists another point $p' \in Z_\Omega$ such that $Ap' = Dp' = q'$.
We can similarly show that, $r = q' = Ap = Dp$ i.e. there exists a unique point of coincidence and so $\delta_\Omega(PC(A,D)) = 0$.
Since, (A, D) is a $\mathcal{JHR}$ operator so there exists a sequence $\{t_n\}$ in Z_Ω such that

$$\lim_{n\to\infty} At_n = \lim_{n\to\infty} Dt_n = t$$

and $\lim_{n\to\infty} \|\Omega_\alpha(t_n, t)\| \leq \delta_\Omega(PC(S,T)) = 0$ for all $\alpha > 0$. Clearly, $\lim_{n\to\infty} t_n = t$.

$$(7) \quad \psi(\|\Omega_\alpha(At_n, Ba_{2n+1})\|1_{\mathbb{A}}) \preceq F_*(\psi(M(t_n, a_{2n+1})), \varphi(M(t_n, a_{2n+1})))$$

where,

$$\begin{aligned} M(t_n, a_{2n+1}) = & \|j\|^2\|\Omega_\alpha(Dt_n, Ca_{2n+1})\|1_{\mathbb{A}} + \|L\|^2 min\{\|\Omega_\alpha(At_n, Ca_{2n+1})\|, \\ & \|\Omega_\alpha(Ba_{2n+1}, Dt_n)\|, \|\Omega_\alpha(At_n, Dt_n)\|\}1_{\mathbb{A}} \\ = & \|j\|^2\|\Omega_\alpha(Dt_n, Ca_{2n+1})\|1_{\mathbb{A}} \end{aligned}$$

Let $n \to \infty$, equation (7), equals

$$\begin{aligned} \lim_{n\to\infty} \psi(\|\Omega_\alpha(At_n, Ba_{2n+1})\|1_{\mathbb{A}}) \preceq & \lim_{n\to\infty} F_*(\psi(\|j\|^2\|\Omega_\alpha(Dt_n, Ca_{2n+1})\|1_{\mathbb{A}}), \\ & \varphi(\|j\|^2\|\Omega_\alpha(Dt_n, Ca_{2n+1})\|1_{\mathbb{A}})) \\ \Rightarrow \psi(\|\Omega_\alpha(t, r)\|1_{\mathbb{A}}) \preceq & F_*(\psi(\|j\|^2\|\Omega_\alpha(t, r)\|1_{\mathbb{A}}), \varphi(\|j\|^2\|\Omega_\alpha(t, r)\|1_{\mathbb{A}})) \\ \preceq & F^*(\psi(\|\Omega_\alpha(t, r)\|1_{\mathbb{A}}), \varphi(\|\Omega_\alpha(t, r)\|1_{\mathbb{A}})) \\ \preceq & \psi(\|\Omega_\alpha(t, r)\|1_{\mathbb{A}}) \end{aligned}$$

Similarly as above by the definition of C_*-class function we have, $t = r$. Since D is $\Omega-$continuous and $\lim_{n\to\infty} Dt_n = t = r$ so $Dr = r$.

$$(8) \qquad \psi(\|\Omega_\alpha(Ar, Ba_{2n+1})\|1_{\mathbb{A}}) \preceq F_*(\psi(M(r, a_{2n+1})), \varphi(M(r, a_{2n+1})))$$

where,

$$\begin{aligned} M(r, a_{2n+1}) = \|j\|^2\|\Omega_\alpha(Dr, Ca_{2n+1})\|1_{\mathbb{A}} + \|L\|^2 min\{\|\Omega_\alpha(Ar, Ca_{2n+1})\|, \\ \|\Omega_\alpha(Ba_{2n+1}, Dr)\|, \|\Omega_\alpha(Ar, Dr)\|\}1_{\mathbb{A}} \\ \to \theta \ as \ n \to \infty \end{aligned}$$

Hence, by $n \to \infty$, equation (8) equals

$$\begin{aligned} \psi(\|\Omega_\alpha(Ar, r)\|1_{\mathbb{A}}) = \theta \\ \Rightarrow Ar = r \end{aligned}$$

Since, $A(Z_\Omega) \subseteq C(Z_\Omega)$ so there exists a point $u \in Z_\Omega$ such that $Cu = Ar = r$.

$$(9) \qquad \psi(\|\Omega_\alpha(Ar, Bu)\|1_{\mathbb{A}}) \preceq F_*(\psi(M(r, u)), \varphi(M(r, u)))$$

where,

$$\begin{aligned} M(r, u) = \|j\|^2\|\Omega_\alpha(Dr, Cu)\|1_{\mathbb{A}} + \|L\|^2 min\{\|\Omega_\alpha(Ar, Cu)\|, \\ \|\Omega_\alpha(Bu, Dr)\|, \|\Omega_\alpha(Ar, Dr)\|\}1_{\mathbb{A}} \end{aligned}$$

Thus equation (9) equals

$$\begin{aligned} \psi(\|\Omega_\alpha(r, Bu)\|1_{\mathbb{A}}) = \theta \\ \Rightarrow Bu = r \end{aligned}$$

Since, the pair (B, C) is weakly compatiable and $Bu = Cu = r$. So, $Br = BCu = CBu = Cr$. Now,

$$(10) \qquad \psi(\|\Omega_\alpha(Ar, Br)\|1_{\mathbb{A}}) \preceq F_*(\psi(M(r, r)), \varphi(M(r, r)))$$

where,

$$\begin{aligned} M(r, r) = \|j\|^2\|\Omega_\alpha(Dr, Cr)\|1_{\mathbb{A}} + \|L\|^2 min\{\|\Omega_\alpha(Ar, Cr)\|, \\ \|\Omega_\alpha(Br, Dr)\|, \|\Omega_\alpha(Ar, Dr)\|\}1_{\mathbb{A}} \\ = \|j\|^2\|\Omega_\alpha(r, Br)\|1_{\mathbb{A}} \end{aligned}$$

Thus equation (10) equals

$$\begin{aligned} \psi(\|\Omega_\alpha(r, Br)\|1_{\mathbb{A}}) &\preceq F_*(\psi(\|j\|^2\|\Omega_\alpha(r, Br)\|1_{\mathbb{A}}), \varphi(\|j\|^2\|\Omega_\alpha(r, Br)\|1_{\mathbb{A}})) \\ &\preceq F_*(\psi(\Omega_\alpha(r, Br)\|1_{\mathbb{A}}), \varphi(\|\Omega_\alpha(r, Br)\|1_{\mathbb{A}})) \\ &\preceq \psi(\Omega_\alpha(r, Br)\|1_{\mathbb{A}}) \\ \Rightarrow Br = r \end{aligned}$$

Clearly, $Ar = Dr = Br = Cr = r$. So r is the common fixed point A, B, C and D, and uniqueness of the common fixed point r comes automatically in the similar manner. □

Example 3.1.1 Let $(Z, \mathbb{A}, \Omega)$ be a Ω- complete C^* modular metric space defined as in Example 2.6. Define $A, B, C, D : Z_\Omega \to Z_\Omega$ by

$$Aa = Ba = 1, \qquad Da = 2 - a, \qquad Ca = \begin{cases} \frac{2a}{3} & if\ a \in (-\infty, 1) \\ 1 & if\ a\ = 1 \\ 0 & if\ a \in (1, \infty) \end{cases}$$

Suppose,

$$\begin{cases} \psi : \mathbb{A}_+ \to \mathbb{A}_+ \\ \psi(B) = 2B \end{cases} \quad \begin{cases} \varphi : \mathbb{A}_+ \to \mathbb{A}_+ \\ \varphi(B) = B \end{cases} \quad \begin{cases} F_* : \mathbb{A}_+ \times \mathbb{A}_+ \to \mathbb{A} \\ F_*(A, B) = A - B \end{cases}$$

Then (ψ, φ, F_*) is strictly monotone. For all $a, b \in Z_\Omega = \mathbb{R}$ and $\alpha > 0$, we have

$$0 = \left\| \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\| = \|\Omega_\alpha(Aa, Bb)\| < \infty.$$

For every $j, L \in \mathbb{A}$ with $\|j\| < 1, \|L\| > 0$, we have

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \psi(\|\Omega_\alpha(Aa, Bb)\| 1_{\mathbb{A}}) \preceq F_*(\psi(M(a, b)), \phi(M(a, b))$$

for all $a, b \in Z_\Omega$ and $\alpha > 0$. D is Ω-b-continuous. Also,

$$A(Z_\Omega) = \{1\} \subseteq C(Z_\Omega) = \mathbb{R},\ B(Z_\Omega) = \{1\} \subseteq D(Z_\Omega) = \mathbb{R},$$

$C(A, D) = PC(A, D) = C(B, C) = PC(B, C) = 1$. Suppose $\{a_n\}=\frac{n}{n+1}$ is a b-sequence in Z_Ω where $n = 1, 2, ...$ Then, $\lim_{n\to\infty} Aa_n = Da_n = q = 1$ and

$$\begin{aligned} \lim_{n\to\infty} \|\Omega_\alpha(a_n, z)\| &= \lim_{n\to\infty} \left\| diag(\beta \left| \frac{\frac{n}{n+1} - 1}{\alpha} \right|^2, \beta \left| \frac{\frac{n}{n+1} - 1}{\alpha} \right|^2 \right\| = 0 \\ &\leqslant \delta_\Omega(PC(A, D)) = 0, \text{ for all } \alpha > 0,\ \beta \geqslant 0. \end{aligned}$$

Hence, the pair (A, D) is a $\mathcal{JHR}$ operator pair, and $BC1 = CB1 = 1$ so the pair (B, C) is weakly compatiable. All the conditions of Theorem 3.1 are satisfied so A,B, C and D have a common unique fixed point 1.

Theorem 3.2 Let $(Z_\Omega, \mathbb{A}, \Omega)$ be a Ω-b-complete C^*- algebra valued modular b-metric space with $1_{\mathbb{A}} \preceq K$. Let T be a self-mapping on Z_Ω and satisfies the conditions, for each $a, b \in Z_\Omega$ and $\alpha > 0$:

$$\Omega_\alpha(Ta, Tb) \preceq q^*\Omega_\alpha(a, b)q + min\{L^*\Omega_\alpha(Ta, a)L, L^*\Omega_\alpha(Tb, a)L, L^*\Omega_\alpha(Ta, b)L\}$$

where $q, L \in \mathbb{A}$ with $0 < \|q\| < 1$, $0 < \|L\|$, $\|K\|\|q\|^2 \leqslant 1$, and $\|\Omega_\alpha(Ta, Tb)\| < \infty$.
Then T has a unique fixed point in Z_Ω.

Proof. Let $a_0 \in Z_\Omega$, so there exists a point a_1 such that $a_1 = Ta_0$ hence inductively, we have

$a_{n+1} = Ta_n$ for all $n = 0, 1, 2, \ldots$.

$$\begin{aligned}
\Omega_\alpha(a_{n+1}, a_n) &= \Omega_\alpha(Ta_n, Ta_{n-1}) \\
&\preceq q^*\Omega_\alpha(a_n, a_{n-1})q \\
&\quad + min\{L^*\Omega_\alpha(Ta_n, a_n)L, L^*\Omega_\alpha(Ta_{n-1}, a_n)L, L^*|\Omega_\alpha(Ta_n, a_{n-1})L\} \\
&= q^*(\Omega_\alpha(a_n, a_{n-1}))^{\frac{1}{2}}(\Omega_\alpha(a_n, a_{n-1}))^{\frac{1}{2}}q \\
&\quad + min\{L^*(\Omega_\alpha(Ta_n, a_n))^{\frac{1}{2}}(\Omega_\alpha(Ta_n, a_n))^{\frac{1}{2}}L, \\
&\qquad L^*(\Omega_\alpha(Ta_{n-1}, a_n))^{\frac{1}{2}}(\Omega_\alpha(Ta_{n-1}, a_n))^{\frac{1}{2}}L, \\
&\qquad L^*(\Omega_\alpha(Ta_n, a_{n-1}))^{\frac{1}{2}}(\Omega_\alpha(Ta_n, a_{n-1}))^{\frac{1}{2}}L\} \\
&= (q\Omega_\alpha(a_n, a_{n-1}))^{\frac{1}{2}})^*(\Omega_\alpha(a_n, a_{n-1})^{\frac{1}{2}}q) \\
&\quad + min\{(L\Omega_\alpha(Ta_n, a_n))^{\frac{1}{2}})^*(\Omega_\alpha(Ta_n, a_n)^{\frac{1}{2}}L), \\
&\qquad (L\Omega_\alpha(Ta_{n-1}, a_n))^{\frac{1}{2}})^*(\Omega_\alpha(Ta_{n-1}, a_n)^{\frac{1}{2}}L), \\
&\qquad (L\Omega_\alpha(Ta_n, a_{n-1}))^{\frac{1}{2}})^*(\Omega_\alpha(Ta_n, a_{n-1})^{\frac{1}{2}}L)\} \\
&= |q(\Omega_\alpha(a_n, a_{n-1}))^{\frac{1}{2}}|^2 + min\{|L(\Omega_\alpha(Ta_n, a_n))^{\frac{1}{2}}|^2, \\
&\quad |L(\Omega_\alpha(Ta_{n-1}, a_n))^{\frac{1}{2}}|^2, |L(\Omega_\alpha(Ta_n, a_{n-1}))^{\frac{1}{2}}|^2\} \\
&\preceq \|q(\Omega_\alpha(a_n, a_{n-1}))^{\frac{1}{2}}\|^2 1_{\mathbb{A}} + min\{\|L(\Omega_\alpha(Ta_n, a_n))^{\frac{1}{2}}\|^2 1_{\mathbb{A}}, \\
&\quad \|L(\Omega_\alpha(Ta_{n-1}, a_n))^{\frac{1}{2}}\|^2 1_{\mathbb{A}}, \|L(\Omega_\alpha(Ta_n, a_{n-1}))^{\frac{1}{2}}\|^2 1_{\mathbb{A}}\} \\
&= \|q(\Omega_\alpha(a_n, a_{n-1}))^{\frac{1}{2}}\|^2 1_{\mathbb{A}}
\end{aligned}$$

Thus

$$\|\Omega_\alpha(a_{n+1}, a_n)\| \leqslant \|q\|^2\|\Omega_\alpha(a_n, a_{n-1})\| \leqslant \ldots \leqslant \|q\|^{2n}\|\Omega_\alpha(a_1, a_0)\|$$

Thus for $n > m$

$$\begin{aligned}
\Omega_\alpha(a_m, b_n) &\preceq K\Omega_{\frac{\alpha}{n-m}}(a_m, b_{m+1}) + K^2\Omega_{\frac{\alpha}{n-m}}(b_{m+1}, b_{m+2}) + \ldots + K^{n-m}\Omega_{\frac{\alpha}{n-m}}(b_{n-1}, b_n) \\
\Rightarrow \|\Omega_\alpha(a_m, b_n)\| &\leqslant \|K\Omega_{\frac{\alpha}{n-m}}(a_m, b_{m+1})\| + \|K^2\Omega_{\frac{\alpha}{n-m}}(b_{m+1}, b_{m+2})\| + \ldots \\
&\quad + \|K^{n-m}\Omega_{\frac{\alpha}{n-m}}(b_{n-1}, b_n)\| \\
&\leqslant \|K\|\|q\|^{2m}\|\Omega_{\frac{\alpha}{n-m}}(a_1, a_0)\| + \|K\|^2\|q\|^{2(m+1)}\|\Omega_{\frac{\alpha}{n-m}}(a_1, a_0)\| + \ldots \\
&\quad + \|K\|^{n-m}\|q\|^{2(n-1)}\|\Omega_{\frac{\alpha}{n-m}}(a_1, a_0)\| \\
&= \|K\|\|q\|^{2m}(1 + \|K\|\|q\|^2 + \ldots + (\|K\|\|q\|^2)^{(n-m-1)})\|\Omega_{\frac{\alpha}{n-m}}(a_1, a_0)\| \\
&\leqslant \|K\|\|q\|^{2m}(\sum_{i=0}^{\infty}(\|K\|\|q\|^2)^i)\|\Omega_{\frac{\alpha}{n-m}}(a_1, a_0)\| \\
&= \frac{\|K\|\|q\|^{2m}}{1 - \|K\|\|q\|^2}\|\Omega_{\frac{\alpha}{n-m}}(a_1, a_0)\| \\
&\to 0 \; as \; m \to \infty
\end{aligned}$$

Therefore $\{a_n\}_{n=0}^{\infty}$ is a b-Cauchy sequence and $\lim_{n\to\infty} a_n = p$, say (since Z_Ω is complete). Now,

$$\begin{aligned}\Omega_\alpha(a_n, Tp) &= \Omega_\alpha(Ta_{n-1}, Tp)\\ &\preceq q^*\Omega_\alpha(a_{n-1}, p)q + min\{L^*\Omega_\alpha(Ta_{n-1}, a_{n-1})L, L^*\Omega_\alpha(Tp, a_{n-1})L,\\ &\qquad L^*\Omega_\alpha(Ta_{n-1}, p)L\}\\ \Rightarrow \|\Omega_\alpha(a_n, Tp)\| &\leqslant \|q\|^2\|\Omega_\alpha(a_{n-1}, p)\| \leqslant ... \leqslant \|q\|^{2n}\|\Omega_\alpha(a_0, p)\|\\ &\to 0 \text{ as } n \to \infty\\ \Rightarrow \lim_{n\to\infty} a_n &= Tp = p\end{aligned}$$

Therefore, p is a fixed point of T. If there exists another fixed point $r, (r \neq p)$ such that $Tr = r$, then

$$\begin{aligned}\Omega_\alpha(Tr, Tp) &= \Omega_\alpha(Tr, Tp)\\ &\preceq q^*\Omega_\alpha(r, p)q + min\{L^*\Omega_\alpha(Tr, r)L, L^*\Omega_\alpha(Tp, r)L, L^*\Omega_\alpha(Tr, p)L\}\\ \Rightarrow \|\Omega_\alpha(r, p)\| &\leqslant \|q\|^2\|\Omega_\alpha(r, p)\| \leqslant ... < \|\Omega_\alpha(r, p)\|\end{aligned}$$

which is a contradiction so, $r = p$. Hence, T has a unique fixed point p in Z_Ω. □

Example 3.2.1: Let $Z = l^\infty(S)$ and $H = l^2(S)$, S be a non empty set, and $\mathbb{A} = B(H)$, the set of bounded linear operator on Hilbert space H. Define $\Omega : (0,\infty) \times Z \times Z \to B(H)_+$ by

$$\Omega_\alpha(\{f_n\}, \{g_n\}) = \pi_{\left|\frac{\{f_n\}-\{g_n\}}{\alpha}\right|^p} \text{ for all } \{f_n\}, \{g_n\} \in Z,\ \alpha > 0, p \geqslant 1,$$

where $\pi_h : H \to H$ is the multiplication operator defined by $\pi_h(\phi) = h.\phi$, $\phi \in H$. Then $(Z_\Omega, B(H), \Omega)$ is a Ω-b-complete C^* modular b-metric space with $\|K\| \geqslant 2^p$.

Define $T : Z_\Omega \to Z_\Omega$ by

$T(f_m) = \{\left(\frac{m+1}{m}\right)^2, 0, 0, ...\}$ where $\{f_m\} = \{\frac{m+1}{m}, 0, 0, ...\}$

Suppose

$$\begin{cases}\psi : \mathbb{A}_+ \to \mathbb{A}_+\\ \psi(A) = 2A\end{cases} \quad \begin{cases}\varphi : \mathbb{A}_+ \to \mathbb{A}_+\\ \varphi(A) = A\end{cases} \quad \begin{cases}F_* : \mathbb{A}_+ \times \mathbb{A}_+ \to \mathbb{A}\\ F_*(A, B) = A - B\end{cases}$$

Then (ψ, φ, F_*) is strictly monotone. For all $\{f_m\}, \{f_n\} \in Z_\Omega = l^\infty(S)$, and $\alpha > 0$ we have

$$\begin{aligned}\|\Omega_\alpha(T(\{f_m\}), T(\{f_n\})\| &= \left\|\left|\frac{T\{f_m\} - T\{f_n\}}{\alpha}\right|^p\right\|_\infty\\ &= \left\|\left|\frac{\{(1+\frac{1}{m})^2 - (1+\frac{1}{n})^2, 0, 0...\}}{\alpha}\right|^p\right\|_\infty < \infty;\ n, m \in \mathbb{N}(n \geqslant m).\end{aligned}$$

For every $a, L \in \mathbb{A}$ with $\|a\| = \frac{1}{2^{\frac{p+1}{2}}} < 1$ and $\|L\| \geq 0$ such that $\|K\|\|a\|^2 < 1$, we have

$$\begin{aligned}\pi_{\left|\frac{T\{f_m\}-T\{f_n\}}{\alpha}\right|^p} &= \Omega_\alpha(Tf_m, Tf_n)\\ &\preceq q^*\Omega_\alpha(f_m, f_n)q + min\{L^*\Omega_\alpha(Tf_m, f_m)L, L^*\Omega_\alpha(Tf_n, f_m)L, L^*\Omega_\alpha(Tf_m, f_n)L\}.\end{aligned}$$

Since, it satisfies all the conditions of Theorem 3.2, so T has a unique fixed point $\{1, 0, 0, ...\}$.

4. Applications

In ([18],[19],[20],[21],[22],[23],[24]), applications for the existence and uniqueness results for a system of nonlinear integral equation and on operator equation in fixed point theory can be seen. Here we give such types of applications in C^* algebra valued modular b-metric spaces.

Theorem 4.1 Let H be a Hilbert space and $B(H)$ be the set of all bounded linear operators on H. Let $C_1, C_2, ..., C_n \in B(H)$ with $\sum_{n=1}^{\infty} \|C_n\| \leqslant 1$ and $Q \in B(H)_+$. Then the operator equation

$$Z - \sum_{n=1}^{\infty} C_n^* Z C_n = Q$$

has a unique solution in $\mathbb{A}$.

Proof. Set $\lambda = (\sum_{n=1}^{\infty} \|C_n\|)^p$ with $p \geqslant 1$ and $\|\lambda\| = \frac{1}{2^{\frac{p+1}{2}}} < 1$. Define $\Omega : (0, \infty) \times B(H) \times B(H) \to B(H)_+$, forall $\alpha > 0$, $p \geqslant 1$, $Z_1, Z_2, F \in B(H)_+$ (F is a positive operator) by

$$\Omega_\alpha(Z_1, Z_2) = \left\| \frac{Z_1 - Z_2}{\alpha} \right\|^p F.$$

Then $(B(H), B(H), \Omega)$ is a Ω-b-complete C^* algebra valued modular b-metric space (as in Example 2.6) with $\|K\| = 2^p > 1$. Clearly, $\|K\|\|\lambda\|^2 < 1$ and $\|\Omega_\alpha(Z_1, Z_2)\| < \infty$.
Define a self mapping $T : B(H) \to B(H)$ by

$$T(Z) = \sum_{n=1}^{\infty} C_n^* Z C_n + Q.$$

Then

$$\begin{aligned}
\Omega_\alpha(T(Z_1), T(Z_2)) &= \left\| \frac{T(Z_1) - T(Z_2)}{\alpha} \right\|^p F \\
&= \left\| \frac{\sum_{n=1}^{\infty} C_n^*(Z_1 - Z_2) C_n}{\alpha} \right\|^p F \\
&\preceq \sum_{n=1}^{\infty} \|C_n\|^{2p} \left\| \frac{Z_1 - Z_2}{\alpha} \right\|^p F \\
&\preceq \lambda^2 \Omega_\alpha(Z_1, Z_2) \\
&= (\lambda 1_{\mathbb{A}})^* \Omega_\alpha(Z_1, Z_2)(\lambda 1_{\mathbb{A}}) \\
&\preceq q^* \Omega_\alpha(Z_1, Z_2) q + min\{L^* \Omega_\alpha(TZ_1, Z_1)L, \\
&\qquad L^* \Omega_\alpha(TZ_1, Z_2)L, L^* \Omega_\alpha(TZ_2, Z_1)L\},\ (q = \lambda 1_{\mathbb{A}},\ \|L\| > 0)
\end{aligned}$$

Since all the conditions of Theorem 3.2 are satisfied. So, there exists a unique fixed point $Z \in B(H)_+$, which is a solution of Hermitian operator as $\sum_{n=1}^{\infty} C_n^* Z C_n + Q$ is a positive operator. □

Recall the Example 2.7, Let $Z = L^\infty(E)$ and $H = L^2(E)$, E is a Lebesgue measurable set, and $\mathbb{A} = B(H)$, the set of bounded linear operator on Hilbert space H. Define $\Omega : (0,\infty) \times Z \times Z \to B(H)_+$ by

$$\Omega_\alpha(f,g) = \pi_{|\frac{f-g}{\alpha}|^p} \text{ for all } f,g \in Z,\ \alpha > 0, p \geqslant 1,$$

where $\pi_h : H \to H$. Then $(Z_\Omega, B(H), \Omega)$ is a Ω-b-complete C^* modular b-metric space with $\|K\| \geqslant 2^p$.

4.2 Theorem: Consider the following system of nonlinear integral equations:

$$z(t) = g_i(r, z(r)) + \mu \int_E m(r,s) q_j(s, z(s)) ds + F(r),\ r \in E$$

where E is a Lebesgue measurable set with $m(E) < \infty$, $F \in L^\infty(E)_\Omega$ is known, for $i,j = 1,2,\ i \neq j,\ g_i(r,z(r)), m(r,s), q(s,z(s))$ are complex or real valued functions and measurable both in r and s on E, μ is a complex or real number. Suppose

(i) $sup_{r\in E} \int_E |m(r,s)| ds = N_1 < +\infty$.

(ii) $g_1(s,z)$ is Ω-b-continuous in s and z. For all $s \in E$, $z \in L^\infty(E)_\Omega$ we have $g_i(s,z(s)) \in L^\infty(E)_\Omega$ and there exists $M_1 > 0$ such that

$$\left|\frac{g_1(s,z(s)) - g_2(s,y(s))}{\sqrt{2}}\right|^p \geqslant M_1 |z(s) - y(s)|^p, \text{ for all } z,y \in L^\infty(E)_\Omega, p \geqslant 1.$$

(iii) for all $s \in E$, $z \in L^\infty(E)_\Omega$ we have $q_i(s,z(s)) \in L^\infty(E)_\Omega$ and there exists $M_2 > 0$ such that

$$|q_1(s,z(s)) - q_2(s,y(s))|^p \leqslant M_2 |z(s) - y(s)|^p, \text{ for all } z,y \in L^\infty(E)_\Omega, p \geqslant 1.$$

(iv) let $\mathbf{B}_i \neq \phi, (i = 1,2)$ is a set consists of $z_{p_i} \in L^\infty(E)_\Omega$ such that

$$g_i(r, z_{p_i}(r)) = z_{p_i}(r) - \lambda \int_E m(r,s) q_i(s, z_{p_i}(s)) ds - F(r) = k_{p_i}$$

for any $z_{p_2} \in \mathbf{B}_2$, this implies

$$g_2(r, z_{p_2}(r)) - \mu \int_E m(r,s) q_2(s, z_{p_2}(s)) ds - F(r) = g_2(r, z_{p_2}(r)) - \mu \int_E m(r,s) q_2(s, z_{p_2}(s)) ds - F(r).$$

(v) there exists a sequence $\{z_n(r)\} \in L^\infty(E)_\Omega$ such that

$$\lim_{n\to\infty} g_1(r, z_n(r)) = \lim_{n\to\infty} z_n(r) - \lambda \int_E m(r,s) q_1(s, z_n(s)) ds - F(r) = \beta_1(r) \in L^\infty(E)_\Omega$$

and satisfies $\lim_{n\to\infty} \|\Omega_\alpha(z_n, \beta_1)\| \leqslant \left\|\left|\frac{g_1(r,z_{p_1}) - g_1(r,z_{q_1})}{\alpha}\right|^p\right\|_\infty$ for all $\alpha > 0$ and $z_{p_1}, z_{q_1} \in \mathbf{B}_1$.

Then the system of nonlinear integral equations has a unique solution $z^* \in L^\infty(E)_\Omega$ for each complex or real number μ with $\|K\| = 2^p$.

Proof. Define $A, B, C, D : Z_\Omega \to Z_\Omega$ by

$$Az(r) = z(r) - \mu \int_E m(r,s)q_1(s,z(s))ds - F(r),$$
$$Bz(r) = z(r) - \mu \int_E m(r,s)q_2(s,z(s))ds - F(r),$$
$$Cz(r) = g_2(r,z(r)), Dz(r) = g_1(r,z(r))$$

Set $\|j\| = \sqrt{\frac{1+|\mu|N_1M_2}{M_1}} < 1$ and $\|L\| > 0$. Again Define,

$$\begin{cases}\psi : B(H)_+ \to B(H)_+ \\ \psi(B) = \frac{1}{2}B\end{cases} \quad \begin{cases}\varphi : B(H)_+ \to B(H)_+ \\ \varphi(B) = \frac{1}{4}B\end{cases} \quad \begin{cases}F_* : B(H)_+ \times B(H)_+ \to B(H) \\ F_*(A,B) = \frac{1}{2^{p/2}}A, p \geqslant 1\end{cases}$$

Clearly, (ψ, φ, F_*) is strictly monotonic, and D is Ω-continuous from (ii).

$$\begin{aligned}
\psi(\|\Omega_\alpha(Az,By)\|1_{\mathbb{A}}) &= \frac{1}{2}sup_{\|h\|=1}(\pi_{|\frac{Az-By}{\alpha}|^p h,h})1_{\mathbb{A}} \\
&= sup_{\|h\|=1}\int_E [\frac{1}{2\alpha^p}|(z-y) \\
&\quad + \mu\int_E m(r,s)(q_2(s,g(s)) - q_1(s,f(s)))ds|^p]h(r)\overline{h(r)}dr1_{\mathbb{A}} \\
&\preceq \frac{1}{2\alpha^p}sup_{\|h\|=1}\int_E |h(r)|^2dr[\||z-y|^p\|_\infty + |\mu|N_1M_2\||f-g|^p\|_\infty]1_{\mathbb{A}} \\
&\preceq \frac{1+|\mu|N_1M_2}{2\alpha^p}\||z-y|^p\|_\infty]1_{\mathbb{A}} \\
&\preceq (\frac{1}{2^{p/2}})(\frac{1}{2})(\frac{1+|\mu|N_1M_2}{M_1})\left\|\left|\frac{g_1(r,z(r)) - g_2(r,y(r))}{\alpha}\right|^p\right\|_\infty 1_{\mathbb{A}} \\
&= (\frac{1}{2^{p/2}})(\frac{1}{2})\|j\|^2\,\|\Omega_\alpha Dz, Ay\|\,1_{\mathbb{A}} \\
&\preceq (\frac{1}{2^{p/2}})(\frac{1}{2})(\|j\|^2\,\|\Omega_\alpha Dz, Ay\|\,1_{\mathbb{A}} \\
&\quad + \|L\|^2 min\{\|\Omega_\alpha(Az,Cy)\|, \|\Omega_\alpha(By,Dz)\|, \|\Omega_\alpha(Az,Dz)\|\}1_{\mathbb{A}})
\end{aligned}$$

Thus,

$$\psi(\|\Omega_\alpha(Az,By)\|1_{\mathbb{A}}) \preceq F_*(\psi(M(z,y)), \varphi(M(z,y)))$$

By condition (iv), $\mathbf{B}_1 = C(A,D) \neq \phi, \mathbf{B}_2 = C(A,D) \neq \phi$ and

$$Cz_{p_2} = Bz_{p_2} \Rightarrow B(Cz_{p_2}) = C(Bz_{p_2}),$$

so (B,C) is a weakly compatible.
From (iv) and (v), $PC(A,D) \neq \phi$ and there exists a sequence $\{z_n(r)\} \in L^\infty(E)_\Omega$ such that

$$\lim_{n\to\infty} Az_n = \lim_{n\to\infty} Dz_n = \beta_1$$

and satisfies

$$\begin{aligned}\lim_{n\to\infty}\|\Omega_\alpha(z_n,\beta_1)\| &\leqslant \left\|\left|\frac{g_1(r,z_{p_1})-g_1(r,z_{q_1})}{\alpha}\right|^p\right\|_\infty, \text{for all } \alpha>0 \text{ and } z_{p_1},z_{q_1}\in\mathbf{B}_1\\ &=\|\Omega_\alpha(Dz_{p_1},Dz_{q_1}))\|, \text{for all } \alpha>0 \text{ and } z_{p_1},z_{q_1}\in\mathbf{B}_1\\ &=\|\Omega_\alpha(Az_{p_1},Az_{q_1}))\|, \text{for all } \alpha>0 \text{ and } z_{p_1},z_{q_1}\in\mathbf{B}_1\\ &\leqslant \sup_{z,y\in PC(A,D)}\|\Omega_\alpha(z,y)\|, \text{for all } \alpha>0\\ &=\delta_\Omega(PC(A,D))\end{aligned}$$

Hence, (A,D) is a $\mathcal{JHR}$-operator pair. So, by Theorem 3.1, there exists a unique common fixed point $z^*\in L^\infty(E)_\Omega$, which proves the existence of unique solution of the non linear integral equation in $L^\infty[0,1]_\Omega$. □

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